OBSTRUCTIONS FOR UNIFORM STABILITY OF C_0-SEMIGROUP

K. V. STOROZHUK

ABSTRACT. Let $T_t : X \to X$ be a C_0-semigroup with generator A. We prove that if the abscissa of uniform boundedness of the resolvent $s_0(A) \geq 0$ then for each a non-decreasing function $h(s) : \mathbb{R}_+ \to R_+$, there are $x' \in X'$ and $x \in X$ such that $\int_0^\infty h(|\langle x', T_t x\rangle|)dt = \infty$. If $i\mathbb{R} \cap Sp(A) \neq \emptyset$ then such x may be taken in $D(A^\infty)$.

1. Introduction and preliminary results

Let X be a Banach space. C_0-semigroup $T_t : X \to X$ is called uniformly exponentially stable (UES), if $\|T_t x\|$ decays exponentially with t for all $x \in X$. By the Uniform Boundedness Principle (UBP) it is equivalent to $\|T_t\| \to_{t\to\infty} 0$.

In the finite-dimensional case, these conditions are equivalent to the fact that $\|T_t x\| \to 0$ as $t \to \infty$ for all $x \in X$. The basic infinite-dimensional counterexample is given by the semigroup of shifts on $L_2(\mathbb{R}_+)$. Here $\|T_t\| \equiv 1$ for all t, but $\|T_t x\| \to 0$ for all x. However, the absence of (UES) for a semigroup implies the existence of the vectors whose orbits, while tending to zero, do it "very slowly". In this connection, we mention the articles [1, 2] which, in particular, imply that if the spectral radius $r(T) = 1$, then, for every sequence $1 > \alpha_n \to 0$, there exists $x \in X$ such that $\|T^n x\| > \alpha_n$ for all n.

For C_0-semigroups, let us give a typical result demonstrating the absence of the upper estimate for the decrease rate "in the integral sense".

Theorem 0. *Suppose that $\forall t \geq 0$ $\|T_t\| \geq 1$. For any non-decreasing function $h : \mathbb{R}_+ \to R_+$ ("good" function), there exists $x \in X$ such that $\int_0^\infty h(\|T_t x\|)dt = \infty$. Moreover, if the semigroup T_t is unbounded then for some $x \in X$ $\|T_t x\| > 1$ for t from a set of infinite measure.*

Analogous results hold also for a more general situation of evolution operators $U(t,s)U(s,r) = U(t,r) : X \to X$. There are several proofs of such results going back to [3, 4, 5]. In a short proof the first part belongs to the author [6], the second part is quite analogous to the beginning of the proof of the Theorem 3.2.2 in [7]. We will present this proof, first, because of its brevity, and, second, because we will compare other reasoning with it.

Proof of Theorem 0. The following "backward" estimate holds for a bounded C_0-semigroup:

$$\text{Let } C = \sup_{t>0}\{\|T_t\|\}. \text{ Then } \forall t_0\ \forall t \in [0,t_0],\ \forall x \in X\ \|T_t x\| \geq \frac{\|T_{t_0}x\|}{C} \qquad (*)$$

Choose a sequence $\alpha_n \to 0$ that decreases so slowly that $nh(\alpha_n) \to \infty$. We have $\lim \frac{\|T_n\|}{\alpha_n} = \infty$. Following the UBP, choose $x \in X$ such that $\overline{\lim_{n\to\infty}} \frac{\|T_n x\|}{\alpha_n} \geq C$. Then

$$\int_0^\infty h(\|T_t x\|)dt = \sup_n \int_0^n h(\|T_t x\|)dt \overset{(*)}{\geq} \sup_n \int_0^n h\left(\frac{\|T_n x\|}{C}\right) dt \geq \sup_n n \cdot h(\alpha_n) = \infty.$$

Now, let $\|T_t\|$ be an unbounded C_0-semigroup. In this case, we have a weaker analog of $(*)$, a “finite-time backward estimate”. For example,

$$\text{Let } \ C = \sup_{t\in[0,1]} \{\|T_t\|\}. \text{ Then } \ \forall t_0\ \forall t \in [t_0 - 1, t_0],\ \forall x \in X\ \|T_t x\| \geq \frac{\|T_{t_0} x\|}{C} \qquad (**)$$

The estimate $(**)$ is weaker than the estimate $(*)$. On the other hand, according to UBP, for the unbounded semigroup there exists $x \in X$ whose orbit $T_t x$ does not tend to zero. Let $\|T_{n_k} x\| > C$, $n_k \to \infty$. Formula $(**)$ implies that $\|T_t x\| > 1$ with t in the set $\cup_{k=1}^\infty [n_k - 1, n_k]$.

The questions, concerning the orbits' slow approaching to zero in the weak topology of the space X, appeared for the first time and started to be discussed in [8, 9, 10]. A review of this subject can be found in [7]. A question, which is analogous to the conclusion of Theorem 0 for the weak topology, is the following: under which assumptions on the semigroup one can claim that for each "good" function h

$$\exists x \in X,\ x' \in X' \ \int_0^\infty h(|\langle x', T_t x\rangle|)\, dt = \infty. \tag{1}$$

The absence of UES alone is not sufficient here, as can be shown by the example (see [11]; [12], example 1.5) of the semigroup of shifts on $L^1(\mathbb{R}_+, e^t dt) \cap L^p(\mathbb{R}_+)$. This semigroup is not UES but is weakly L^1-stable; i.e.,

$$\forall x \in X,\ x' \in X' \ \int_0^\infty |\langle x', T_t x\rangle|\, dt < \infty.$$

From the standpoint of geometry, this example looks rather surprising: some orbits are "far" from zero; at the same time, the orbit of each vector x spends almost all the time "arbitrarily close" to each hyperplane ($\ker x'$)!

It is known that the conclusion (1) holds, for example, for the bounded C_0-semigropus, if one requires the absence of not only the *uniform exponential stability* but simply the *exponential stability.* (see the beginning of the next section and Proposition 1).

The main result of the present article (Theorem 1) is the estimates from which it follows, in particular, that, in the Proposition 1, the condition of the semigroup being bounded is dispensable.

In the next section we briefly discuss some asymptotic parameters of the semigroup which are necessary for precise formulation of the results and formulate the appropriate statements. The third section is a proof of the basic result. In the fourth section we prove that, under some natural assumptions, the vector x in the formula (1) can be chosen among "smooth" vectors.

2. Asymptotic parameters and the basic results

Uniform growth bound of semigroup is

$$\omega_0(T) = \lim_{t\to\infty} \frac{\ln \|T_t\|}{t} = \sup\{\overline{\lim}_{t\to\infty} \frac{\ln \|T_t x\|}{t} \mid x \in X\}.$$

Let A be a generator of the semigroup and let $D(A)$ be a domain of A. The growth bound of semigroup is

$$\omega_1(T) := \sup\{\overline{\lim}_{t\to\infty} \frac{\ln \|T_t x\|}{t} \mid x \in D(A)\}.$$

Let us explain informally the meaning of ω_0 and ω_1. Consider an abstract Cauchy problem $\frac{dz}{dt} = Az,\ z(0) = x \in X$. The function $z(t) = T_t x$ is called a *mild solution* of ACP. If the initial value x lies in $D(A)$, then it is natural for the solution to be called *classical*, or *smooth* solution (accordingly, such initial values x from $D(A)$ are called the smooth vectors). So, the parameters ω_0 and ω_1 describe the growth of the mild (accordingly, smooth) solutions of our ACP.

Clearly, $\omega_1 \le \omega_0$. It is easy to see that a semigroup is UES if and only if $\omega_0(T) < 0$. If $\omega_1(T) < 0$, then the semigroup is simply called exponentially stable, or (ES)

For the semigroup of shifts, mentioned in the end of the introduction, $\omega_1 < 0 = \omega_0$; therefore though it is not UES but, nevertheless, it is ES.

Theorem 0 can still be "rescued" for the weak topology also, if, for example, one requires that the semigroup, together with being not UES, is not ES.

Proposition 1 *(see* [7]*, Theorems 4.6.3(i) and 4.6.4). If a uniformly bounded C_0-semigroup $T_t : X \to X$ is not exponentially stable, then the following holds:*

(1) for any "good" $h > 0$, there is $x \in X$ such that $\int_0^\infty h(|\langle x', T_t x\rangle|)dt = \infty$;

(2) there exists an $\varepsilon > 0$ such that for each $m > 0$ there exist norm-one vectors $x \in X$ and $x' \in X'$ such that $m < mes\{t \mid |\langle x', T(t)x\rangle| \ge \varepsilon\}$.

Let us describe two more spectral characteristics, between which the growth ω_1 of the semigroup is "confined". It will be necessary for us to formulate of basic results of article and strengthen Proposition 1. It is the spectral bound $s(A)$ and the abscissa $s_0(A)$ of uniform boundedness of the resolvent of A. We have the following diagram of inequalities $\begin{matrix} \omega_1 & \leftarrow & \omega_0 \\ \downarrow & \nwarrow & \downarrow \\ s & \leftarrow & s_0 \end{matrix}$ (here "$\leftarrow$" stands for "$\le$").

In Hilbert spaces, $s_0 = \omega_0$ ([13]). For positive semigroups, $s = \omega_1 = s_0$; moreover, s is reached by $\sigma(A)$, i.e., there exists $\lambda \in \sigma(A),\ re(\lambda) = s$ ([11, 14, 15]).

On the other hand, for each arrow, there is an example of strict inequality. The first historically example of Foiaş' [16], in which $-2 = s < s_0 = \omega_0 = 0$ is known rather badly. In the foundation of the majority of other such examples there lies an example due to Zabczyk [17]. (Note that Zabczyk himself refers to Foyaş in the text of the paper [17]). In Zabczyk's example $s < \omega_1 = \omega_0$; s is reached and $\|T_t\| = e^{t\omega_0}$. These and other examples can be found in [14] and [7].

It turns out that the conclusion of Proposition 1 is valid also for non-bounded semigroups, whereas the condition of "being not ES" ($\omega_1(T) \ge 0$) can be substituted by a weaker condition "$s_0(A) \ge 0$". This follows from our main result:

Theorem 1. *Let A be the generator of C_0-semigroup $T_t : X \to X$ and let $s_0(A) \geq 0$. For any two sequences $0 < m_1 < m_2 < \cdots$ and $\gamma_k > 0$, $\gamma_k \to 0$ there is $x' \in X'$ $x \in X$ and a family of sets $U_k \subset \mathbb{R}_+$ such that*

$$\forall k \in \mathbb{N}\ \mu(U_k) \geq m_k,\ \forall t \in U_k\ \ |\langle x', T_t x\rangle| > \gamma_k.$$

To derive Proposition 1 from Theorem 1, it suffices to find, for the function h, the numbers m_k growing so rapidly that $m_k \cdot h(\gamma_k) \to \infty$, in which case the integral in Proposition 1 diverges (compare with the proof of Theorem 0).

Notice. In fact, the proof of Theorem 4.6.3 of [7] uses only the condition on s_0 instead of ω_1. However, the assumption of boundedness is essential for those methods (the proof in [7] is based on the technique of rearrangement-invariant Banach function spaces). In our proof of Theotem 1 we will need some methodological and technical tricks which would be unnecessary under the assumption of the boundedness of the semigroup.

3. Proof of Theorem 1

Our proof of Theorem 1 is based on the following lemma.

Lemma 1. *Suppose that $s_0(A) = 0$. Then, for all $\delta > 0$ and $t_0 < \infty$ there are $\beta = \beta(\delta, t_0) \in \mathbb{R}$ and $y \in D(A)$ such that $\|y\| = 1$, $\|Ay\| \approx |\beta|$ and $\|T_t y - e^{i\beta t}y\| < \delta$ for all $t \in [0, t_0]$. Moreover, such y may be chosen in $D(A^\infty) = \cap_n D(A^n)$.*

Notice. Geometrically, Lemma 1 means that there is a unit vector y staying near the (complex) line $\mathbb{C}y$ for a long time (at that, this vector is "spinning" in the corresponding real plane with angular velocity β). At the same time, the assertion (2) of Proposition 1 means merely that there is a unit vector x staying away from the hyperplane ker x' for a long time. So, (2) (proved in [7] with the use of assertion (1)) follows already from Lemma 1.

Proof of Lemma 1. Note that, given a semigroup T_t with generator A, we have

$$\forall x \in D(A)\ \forall t > 0, \beta \in \mathbb{R}\ \ \|T_t x - e^{i\beta t}x\| \leq t \cdot \sup_{s\in[0,t]} \|T_s\| \cdot \|(A - i\beta)x\|. \qquad (2)$$

Indeed, $(A - i\beta)$ is the generator of the semigroup $e^{-i\beta t}T_t$, so $\|T_t x - e^{i\beta t}x\| = \|e^{-i\beta t}T_t x - x\| = \|\int_0^t T_s(A - i\beta)x\,ds\|$ etc.

As $s_0(A) = 0$, the resolvent $(A - \lambda)^{-1}$ of operator A is unbounded if λ is near the imaginary axis. Choose $\lambda_n = \alpha_n + i\beta_n \in \mathbb{C}$, $\alpha_n \to 0$, $\|(\lambda_n - A)^{-1}\| \to \infty$. Take the vectors $y_n \in D(A)$ such that $\|y_n\| = 1$ but $(A - \lambda_n)y_n \to 0$. In particular, $(A - i\beta_n)y_n \to 0$. Take n so large that $\|(A - i\beta_n)y_n\| < \frac{\delta}{t_0 \cdot \sup_{s\in[0,t]} \|T_s\|}$.

The set $D(A^\infty)$ is dense in $D(A)$ in the graph norm ([14], 1.9(iii)), so such y may be chosen in $D(A^\infty)$. Now, involving (2), we finish the proof of the lemma.□

Proof of Theorem 1. Clearly, if there are $x' \in X'$, $x \in X$, and $\delta > 0$ such that $mes\{t > 0 \mid |\langle x', T_t x\rangle| > \delta\} = \infty$, then there is nothing to prove. In particular, we may assume from the very beginning that

$$\forall x \in D(A)\ \forall x' \in X'\ \forall \delta > 0\ \ mes\{t > 0 \mid |\langle x', T_t x\rangle| > \delta\} < \infty. \qquad (3)$$

Suppose that X is separable, this will enable us to use the sequential compactness of X' in the $*$-weak topology. It does not restrict the generality: it is not difficult to see that in general case Theorem 1 can be applied to an appropriate separable subspace $X_s \subset X$, invariant under the action of the semigroup, and then continue the corresponding functional $x'_s \in (X_s)'$ to a functional x' on the entire X.

Last, note that it is enough for us to prove the theorem for any concrete sequence $\gamma'_k \to 0$. (Let us explain why it does not restrict the generality either. Let $m_k \in \mathbb{N}$ and $\gamma_k \to 0$. We may suppose that $\gamma_k < \gamma'_1$ for all k. Put $n(k) = \max\{n \mid \gamma_n \geq \gamma'_k\}$ and $m'_k = m_1 + \cdots + m_{n(k)}$. It is easy to see that if x, x' satisfy the hypothesis of Theorem 1 with m'_k and γ'_k then they will also satisfy of Theorem 1 for initial m_k, γ_k.)

We prove Theorem 1 for $\gamma_k = \frac{5}{10^{2^k-1}}$.

Following Lemma 1, choose a sequence of numbers $\beta_n \in \mathbb{R}$ and vectors $y_n \in D(A)$ such that $\|y_n\| = 1$ and $\|T_t y_n - e^{i\beta_n t} y_n\| \leq \frac{1}{10}$ for some $\beta_n \in \mathbb{R}$ and all $t \in [0, n]$.

For each y_n, choose any dual $y'_n \in X'$, $\|y'_n\| = \langle y'_n, y_n\rangle = 1$. The sequence y'_n contains a subsequence that *-weakly converges to some $y' \in X'$. We may consider $y'_n \overset{\sigma^*}{\to} y'$. It easy to see that

$$\forall t \in [0, n]\ \frac{9}{10} < \|T_t y_n\| < \frac{11}{10},\ \frac{9}{10} < |\langle y'_n, T_t y_n\rangle| < \frac{11}{10} \tag{4}$$

We shall construct x and x' as the limits of x_k and x'_k, $x_k = \sum_{l=1}^k \frac{\pm y_{n_l}}{10^{2^{l-1}-1}}$, $x'_k = \sum_{l=1}^k \frac{\pm y'_{n_l}}{10^{2^{l-1}-1}}$, where the numbers n_l and the signs $\pm$ are to be found.

The construction of the vector x_1. Let $n_1 \in \mathbb{N}$, $n_1 \geq m_1$. Put $U_1 = [0, n_1]$. Put $x'_1 := y'_{n_1}$, $x_1 := y_{n_1}$. Then

$$\forall t \in U_1\ |\langle x'_1, T_t x_1\rangle| > \frac{9}{10} \tag{5}$$

The construction of the vector x_2.

The assumption, expressed by the formula (3), allows choosing a sufficiently large but compact set $\tilde{U}_2$ such that $\mu(\tilde{U}_2) \geq 4m_2$ and

$$\forall t \in \tilde{U}_2\ |\langle y', T_t x_1\rangle| < 1. \tag{6}$$

Choose a number $n_2 \geq n_1$ such that $\tilde{U}_2 \subset [0, n_2]$.

Put $x_2 = x_1 \pm \frac{y_{n_2}}{10}$, $x'_2 = x'_1 \pm \frac{y'_{n_2}}{10}$. We will decide later which pair of signs $\pm$ to choose from the four possible cases. Now, we show that

$$\forall t \in U_1\ |\langle x'_2, T_t x_2\rangle| > \frac{9}{10} - \frac{3}{10}. \tag{7}$$

We have $\langle x'_2, T_t x_2\rangle = \langle x'_1, T_t x_1\rangle \pm \langle x'_1, \frac{T_t y_{n_2}}{10}\rangle \pm \langle \frac{y'_{n_2}}{10}, T_t x_2\rangle$ for any t. Therefore,

$$|\langle x'_2, T_t x_2\rangle| \geq |\langle x'_1, T_t x_1\rangle| - \frac{S_1(t)}{10},\quad S_1(t) = \left(|\langle x'_1, T_t y_{n_2}\rangle| + |\langle y'_{n_2}, T_t x_2\rangle|\right). \tag{8}$$

If $t \in U_1$ then $S_1(t) < 3$. Indeed, if $t \in U_1$ then $|\langle x'_1, T_t y_{n_2}\rangle| \leq \|T_t y_{n_2}\| \leq \frac{11}{10}$ and $|\langle y'_{n_2}, T_t x_2\rangle| \leq \|T_t x_2\| = \|T_t(y_1 \pm \frac{y_{n_2}}{10})\| \leq \frac{11}{10} + \frac{11}{10^2}$. Recalling (5) yields (7).

If $t \notin U_1$ then (8) is useless for the estimation of $|\langle x'_2, T_t x_2\rangle|$ from below because, for example, we cannot estimate the value of $|\langle x'_1, T_t x_1\rangle|$ from below. Let's utilize another trick will be called "choosing from the four".

Note that $2\frac{y'_{n_2}}{10} = (x'_1 + \frac{y'_{n_2}}{10}) - (x'_1 - \frac{y'_{n_2}}{10})$ and, analogously, $2\frac{y_{n_2}}{10} = (x_1 + \frac{y_{n_2}}{10}) - (x_1 - \frac{y_{n_2}}{10})$. Therefore,

$$\forall t\ \ 4\left|\langle \frac{y'_{n_2}}{10}, T_t \frac{y_{n_2}}{10}\rangle\right| \leq \sum_{\pm \in \{+,-\}} \left|\langle (x'_1 \pm \frac{y'_{n_2}}{10}), T_t(x_1 \pm \frac{y_{n_2}}{10})\rangle\right|,$$

and for each $t \in \tilde{U}_2$ at least one of the four terms on the right is at least $\frac{1}{10^2}|\langle y'_{n_2}, T_t y_{n_2}\rangle|$. At the same time, by (4), for all $t \in \tilde{U}_2 \subset [0, n_2]$ $|\langle y'_{n_2}, T_t y_{n_2}\rangle| > \frac{9}{10}$. Therefore, we can choose a subset $U_2 \subset \tilde{U}_2$ whose measure $\mu(U_2) \geq \frac{1}{4}\mu(\tilde{U}_2) \geq m_2$ and for some pair of signs $\pm$ (say, for "++") we have

$$\forall t \in U_2 \ \left|\langle (x'_1 + \frac{y'_{n_2}}{10}), T_t(x_1 + \frac{y_{n_2}}{10})\rangle\right| = |\langle x'_2, T_t x_2\rangle| > \frac{9}{10^3}. \tag{9}$$

The construction of the vector x_3. Let $\tilde{U}_3$ be a compact set, $\mu(\tilde{U}_3) \geq 4m_3$, $\forall t \in \tilde{U}_3$ $|\langle y', T_t x_2\rangle| < 1$. Return for a while to the set U_2. From (6), the compactness of the set $\{T_t x \mid t \in \tilde{U}_2\} \subset X$ and the fact that $y'_n \xrightarrow{\sigma^*} y'$ it follows that

$$\exists n_3 \geq n_2 \mid \tilde{U}_3 \subset [0, n_3], \ \forall n \geq n_3, \forall t \in U_2 \ |\langle y'_n, T_t x_1\rangle| < 1. \tag{10}$$

Put $x_3 = x_2 \pm \frac{y_{n_3}}{10^3}$, $x'_3 = x'_2 \pm \frac{y'_{n_3}}{10^3}$. The pair of signs $\pm$ will be chosen later. Now, arguing in the same way as in deriving (8), we obtain:

$$\forall t \ |\langle x'_3, T_t x_3\rangle| \geq |\langle x'_2, T_t x_2\rangle| - \frac{S_2(t)}{10^3}, \ S_2(t) = |\langle x'_2, T_t y_{n_3}\rangle| + |\langle y'_{n_3}, T_t x_3\rangle|. \tag{11}$$

Show that $\forall t \in U_1 \cup U_2$ $S_2(t) < 3$. Clearly, $\|x'_2\| \leq \frac{11}{10}$. Arguing as in estimating $S_1(t)$, we have: $\forall t \in U_1$ $S_2(t) \leq \left(\frac{11}{10}\right)^2 + \frac{11}{10} + \frac{11}{10^2} + \frac{11}{10^4} < 3$.

If $t \in U_2$ then the argument is a bit more difficult. The first summand in $S_2(t)$ is estimated in the old way: if $t \in U_2 \subset [0, n_3]$, then $\|T_t y_{n_3}\| \leq \frac{11}{10}$ and $|\langle x'_2, T_t y_{n_3}\rangle| \leq (\frac{11}{10})^2$. Consider the summand $|\langle y'_{n_3}, T_t x_3\rangle|$. Recall that $T_t x_3 = (T_t x_1 + \frac{T_t y_{n_2}}{10} \pm \frac{T_t y_{n_3}}{10^3})$.

The value $\|\frac{T_t y_{n_2}}{10} \pm \frac{T_t y_{n_3}}{10^3}\|$ with $t \in U_2$ is less than $\frac{11}{10^2} + \frac{11}{10^4}$.

The vector $T_t x_1 = T_t y_{n_1}$, which for $t \in U_2$ can a priori become large in norm, could have ruined everything but, owing to (10), the value of the y'_{n_3} at this vector at $t \in U_2$ is less than 1. Therefore, $\forall t \in U_2$ $\left|\langle y'_{n_3}, T_t x_3\rangle\right| \leq \left(1 + \frac{11}{10^2} + \frac{11}{10^4}\right)$. Finally, $\forall t \in U_2$ $S_2(t) < \left(\frac{11}{10}\right)^2 + \left(1 + \frac{11}{10^2} + \frac{11}{10^4}\right) < 3$.

Now we conclude from (11), (7), (9) that

$$\forall t \in U_1 \ |\langle x'_3, T_t x_3\rangle| > \frac{9}{10} - \frac{3}{10} - \frac{3}{10^3}, \quad \forall t \in U_2 \ |\langle x'_3, T_t x_3\rangle| > \frac{9}{10^3} - \frac{3}{10^3}.$$

Choose of the pair "$\pm$" and the set $U_3 \subset \tilde{U}_3$, $\mu(U_3) \geq m_3$ with the help of the "choosing from the four" trick. We have:

$$\forall t \in U_3 \ |\langle x'_3, T_t x_3\rangle| > \frac{9}{10} \cdot \left(\frac{1}{10^3}\right)^2 = \frac{9}{10^7}.$$

Note that it was quite easy to construct the vector x_1. At the step 2 we need the "choosing from the four" trick. The novelty of the step 3 was the usage of (10) and the preparation for it — the formula (6) — should be made at the very beginning of the step 2. Next steps have no significant differences from the step 3.

The construction of the vector x_l, $l \geq 3$. Suppose that we have constructed the sets $U_1, \ldots, U_l$, the numbers $n_1 \leq n_2 \leq \cdots \leq n_l$, the vectors $x_1, \ldots, x_l$ of the form $x_l = \sum_{i=1}^{l} \pm \frac{y_{n_i}}{10^{2^{i-1}-1}}$ and, similarly, $x'_1, \ldots, x'_l$ so that the following properties hold:

1_l) $U_i \subset [0, n_i], \mu(U_i) \geq m_i$, $\quad i = 1, 2, \dots, l$;
2_l) $\forall t \in U_i \ |\langle y', T_t x_{i-1}\rangle| < 1$, $\quad i = 2, \dots, l$;
3_l) $\forall t \in U_i \ \forall n \geq n_{i+1} \ |\langle y'_n, T_t x_{i-1}\rangle| < 1$, $\quad i = 2, \dots, l-1$;
4_l) $\forall t \in U_i \ |\langle x'_l, T_t x_l\rangle| \geq \frac{9}{10^{2^i-1}} - 3\sum_{j=i}^{l-1} \frac{1}{10^{2^j-1}}$, $\quad i = 1, \dots, l-1$;
5_l) $\forall t \in U_l \ |\langle x'_l, T_t x_l\rangle| \geq \frac{9}{10^{2^l-1}}$.

Now we construct a set U_{l+1}, a number n_{l+1}, a vector x_{l+1} and the corresponding x'_{l+1} so that the properties $1_{l+1}) - 5_{l+1})$ also hold.

Choose a compact set $\tilde{U}_{l+1}$ such that $\mu(\tilde{U}_{l+1}) \geq 4m_{l+1}$, $\forall t \in \tilde{U}_{l+1} \ |\langle y', T_t x_l\rangle| < 1$. The possibility of such choice follows from (3).

Choose $n_{l+1} \geq n_l$ such that

$$\tilde{U}_{l+1} \subset [0, n_{l+1}], \ \forall n \geq n_{l+1} \ \forall t \in U_l \ |\langle y'_n, T_t x_{l-1}\rangle| < 1.$$

Such n_{l+1} exists by condition 2_l) (see the argument before (10)). It is condition 3_{l+1}.

Put $x_{l+1} = x_l \pm \frac{y_{n_{l+1}}}{10^{2^l-1}}$, $x'_{l+1} = x'_l \pm \frac{y'_{n_{l+1}}}{10^{2^l-1}}$. Choose a pair of signs $\pm$ and a subset $U_{l+1} \subset \tilde{U}_{l+1}$ using the "choosing from the four" trick. Thus, conditions 1_{l+1}, 2_{l+1} hold as well as 5_{l+1}:

$$\forall t \in U_{l+1} \ |\langle x'_{l+1}, T_t x_{l+1}\rangle| \geq \left|\langle \frac{y'_{n_{l+1}}}{10^{2^l-1}}, \frac{T_t y_{n_{l+1}}}{10^{2^l-1}}\rangle\right| \geq \frac{9}{10} \cdot \left(\frac{1}{10^{2^l-1}}\right)^2 = \frac{9}{10^{2^{l+1}-1}}$$

The last, check the condition 4_{l+1}. For all t

$$|\langle x'_{l+1}, T_t x_{l+1}\rangle| \geq |\langle x'_l, T_t x_l\rangle| - \frac{S_l(t)}{10^{2^l-1}}, \quad S_l(t) = |\langle x'_l, T_t y_{n_{l+1}}\rangle| + |\langle y'_{n_{l+1}}, T_t x_{l+1}\rangle|.$$

It suffices to establish that $S_l(t) < 3$ for all $t \in U_1 \cup \dots \cup U_l$. The first summand $S_l(t)$ is less than $\|x'_l\| \cdot \|T_t y_{n_{l+1}}\| \leq \frac{12 \cdot 11}{100}$. Let us estimate the second summand.

If $t \in U_i \subset [0, n_i]$, then, writing $x_{l+1} = x_{i-1} + \sum_{j=i}^{l+1}(\pm\frac{y_{n_j}}{10^{2^{j-1}-1}})$, we have

$$|\langle y'_{n_{l+1}}, T_t x_{l+1}\rangle| = \left|\left\langle y'_{n_{l+1}}, T_t x_{i-1} + \sum_{j=i}^{l+1}(\pm\frac{T_t y_{n_j}}{10^{2^{j-1}-1}})\right\rangle\right| < |\langle y'_{n_{l+1}}, T_t x_{i-1}\rangle| + \frac{1}{2} < \frac{3}{2}.$$

The inequalities in the previous formula are valid due to the smallness of $\|T_t y_j\|$ for $j \geq i$ (as $t \in U_i \subset [0, n_i]$), and also the condition 3_{l+1}, which has been already proved above (sf. the estimation of $S_2(t)$ with $t \in U_2$.)

Let $x' = \lim x'_k$ and $x = \lim x_k$. Then, by "4_∞", we have

$$\forall t \in U_i \ |\langle x', T_t x\rangle| \geq \frac{9}{10^{2^i-1}} - 3\sum_{j=i}^{\infty} \frac{1}{10^{2^j-1}} > \frac{5}{10^{2^i-1}}.$$

The theorem is proved.

4. On the possibility of choosing a smooth vector x in Theorem 1

Is it possible to choose the vector x to be smooth in Theorem 1? Clearly, it cannot be done if, for example, $\omega_1 < s_0 = 0$: in this case the semigroup is ES; therefore, the orbits of smooth vectors decrease too rapidly. Based on example from [17], Wrobel in [18] constructed a semigroup with $s < \omega_1 < s_0 = \omega_0$, moreover, in his example $\omega_n = 2^{-n}$, where ω_n is the growth of the semigroup on $D(A^n)$. So, this

semigroup, after having been rescaled so that $\omega_1 = 0$, remains unbounded, sytisfies the Theorem 1 but decreases exponentially on any smooth vector (and the more smoothness, the faster decrease). The semigroup from [17] itself, renormed so that $s = -1$, $\omega_1 = s_0 = \omega_0 = 0$, is not even ES; however, ([19]): $\|T_t x\| = O(1/t)$ for all $x \in D(A^{1+\varepsilon})$ $\forall \varepsilon > 0$.

What can impede a vector $x = \sum \gamma_j y_{n_j}$ of Theorem 1 from being in $D(A)$ if the number series $\sum \gamma_i \in \mathbb{R}$ converges and $y_{n_j} \in D^\infty(A)$, $\|y_{n_j}\| = 1$? The answer is simple: the series $\sum \gamma_j A y_{n_j}$ does not have to converge in X, as the norms $\|Ay_{n_j}\| \approx |\beta_{n_j}|$ can grow fast with j (see the proof of the Lemma 1).

Suppose that $s_0 = s = 0$ and the bound of Sp(A) is reached, i.e. there exists $\lambda \in Sp(A)$, $Re\lambda = 0$ (a typical case for C_0-semigroups). Then we may take the numbers $i\beta_n$ in Lemma 1 near this λ (not somewhere "near infinity") and, the norms $\|Ay_{n_j}\| \approx |\beta_{n_j}|$ will be bounded with j. Then the series $\sum \gamma_j A y_{n_j}$ converges. Operator A is closed, so x gets to $D(A)$. Let us prove that it is even possible to find *infinitely* smooth vector x:

Theorem 2. *Suppose that $s(A) \geq 0$ is reached. Then there exist $x' \in X'$, $x \in D(A^\infty)$ satisfying the hypothesis of Theorem 1.*

Proof. Let $\lambda \in Sp(A)$, $Re(\lambda) = s$. After rescaling by $e^{-\lambda}$, we may assume $s = 0 \in Sp(A)$. It suffices to find the elements y_n of the proof of Theorem 1 such that $y_n \in D(A^\infty)$ and $\sum_n \gamma_n A^k y_n$ converges in X for all $k = 0, 1, 2, \ldots$. In the next Lemma we will show that it can be done. This Lemma is of interest on its own.

Lemma 2. *If $0 \in Sp(A)$, then $\forall \delta > 0$ $\forall n \in \mathbb{N}$ $\exists y_n \in D(A^\infty)$ such that*

$$\|y_n\| = 1, \ \forall i \leq n \ \ \|A^i y_n\| < \delta. \tag{12_n}$$

Proof. Let $n = 1$. Operator $(\lambda - A)^{-1}$ is unbounded with λ near 0, so, 12_1) followed by Lemma 1.

For $n > 1$ we use the construction of the scale of associated Sobolev semigroups ([14], A-I 3.5). On the space $D(A^n)$, consider the "iterated" graph norm $\|x\|_n = \|x\| + \|Ax\| + \cdots + \|A^n x\|$. The semigroup $T_t : D(A^{n-1}) \to D(A^{n-1})$ with the generator $A : D(A^n) \to D(A^{n-1})$ is isomorphic to the initial one.

Rewriting (12_1) for this semigroup, we infer that there exists $y_n \in D(A^\infty)$ such that

$$\|y_n\| + \|Ay_n\| + \cdots + \|A^{n-1} y_n\| = 1, \ \|Ay_n\| + \cdots + \|A^n y_n\| < \delta. \tag{13}$$

It follows from the second expression (inequality) of (13) that $\|Ay_n\| < \delta$, $\|A^2 y_n\| < \delta, \ldots,$ $\|A^n y_n\| < \delta$. But then from the first *equality* of (13) we obtain that $\|y_n\|$ is about 1. The rest is obvious. Lemma 3 and Theorem 2 are proved.

Note that Theorem 2 seems quite natural, if we take into account the semigroup of shifts: clearly, a function on $[0, \infty)$ can be made to decrease arbitrary slowly and the infinite differentiability, as a local phenomenon, is not an obstacle here.

Acknowledgments. We express our gratitude to Professor Jan van Neerven for his interest in our work and stimulating questions. In particular, his advice enabled us to substantially strengthen the main results.

We thank Professor Constantin Buse for information that stimulated our interest in the problem.

Our thanks also to Professor Bogdan Sasu who sent me a copy of a rather inaccessible paper [16].

References

[1] Muller, V., Local spectral radius formula for operators in Banach spaces, Czechoslovak Math. J. 38(113) (1988), 4, 726–729.

[2] van Neerven, J. M. A. M., On the orbits of an operator with spectral radius one, Czechoslovak Math. J. 45(120) (1995), 3, 495–502.

[3] R. Datko, Extending a theorem of A.M. Lyapunov to Hilbert space, J. Math. Anal. Appl. 32 (1970) 610–616.

[4] Pazy, A., On the applicability of Lyapunov's theorem in Hilbert space, SIAM J. Math. Anal. 3 (1972), 291–294.

[5] S. Rolewicz, On uniform N-equistability, J. Math. Anal. Appl. 115 (1986) 434–441.

[6] Storozhuk, K. V., On the Rolewicz theorem for evolution operators. Proc. Amer. Math. Soc. 135 (2007), no. 6, 1861–1863.

[7] van Neerven, J. M. A. M., The Asymptotic Behaviour of Semigroups of Linear Operators, Birkhauser, Basel, 1996.

[8] Pritchard, A. J.; Zabczyk, J. Stability and stabilizability of infinite-dimensional systems, SIAM Rev. 23 (1981), no. 1, 25–52.

[9] Huang, Fa Lun, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, Ann. Differential Equations 1 (1985), no. 1, 43–56.

[10] Weiss, G., Weak L^p-stability of a linear semigroups on a Hilbert space implies exponential stability, J.Differential Equations 76 (1988), 2, 269–285.

[11] G. Greiner, J. Voigt, and M. Wolff, On the spectral bound of the generator of semigroups of positive operators, J. Operator Th. 5 (1981), 245–256.

[12] van Neerven, J. M. A. M.; Straub, B.; Weis, L., On the asymptotic behaviour of a semigroup of linear operators. Indag. Math. (N.S.) 6 (1995), 4, 453–476.

[13] L. Gearhart, Spectral theory for contraction semigroups on Hilbert spaces, Trans. Am. Math. Soc. 236 (1978), 385–394.

[14] R. Nagel (ed.), One-parameter Semigroups of Positive Operators, Springer Lect. Notes in Math. 1184 (1986).

[15] F. Neubrander, Laplace transform and asymptotic behavior of strongly continuous semigroups, Houston J. Math. 12 (1986), 549-561.

[16] Foiaş, C, Sur une question de M. Reghis, Analele Univ. Timisoara Ser. Sti. Mat. 11 (1973), 111–114.

[17] J. Zabczyk, A note on C_0-semigroups, Bull. Acad. Polon. Sci. 23 (1975), 895-898.

[18] V. Wrobel, Asymptotic behavior of C_0-semigroups in B-convex spaces, Indiana Univ. Math. J. 38 (1989), 101-114.

[19] Batkai, Andras; Engel, Klaus-Jochen; Pruss, Jan; Schnaubelt, Roland, Polynomial stability of operator semigroups. Math. Nachr. 279 (2006), no. 13-14, 1425–1440.

Sobolev Institute of mathematics SB RAS